Web geometry of a system of n first order autonomous ordinary differential equations

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Abstract

Let $dx_i/dt = f_i(x_1, \cdots, x_n)$, $(i = 1, \cdots, n)$ be a system of n first order autonomous ordinary differential equations. We use E. Cartan's equivalence method to study the invariants of this system under diffeomorphisms of the form $\Phi(t, x_1, \cdots, x_n) = (\varphi_0(t), \varphi_1(x_1), \cdots, \varphi_1(x_1))$.

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1 Introduction

The method of equivalence of E. Cartan (see [1], [3] and [4]) provides a powerful tool for constructing differential invariants which solve the problem of deciding when two geometric objects are really the same up to some preassigned group of coordinate transformations. The solution of this problem in terms of differential invariants goes back to S. Lie, but the essential contributions of Cartan was the construction of an adapted coframe whose structure equations yield differential invariants. The analysis of the structure equations provides classification results and yields a natural way of giving invariant characterizations of the special models. In [2] R.B. Gardner gave some examples of solving these problems. For example, he give the local equivalence problem for $dy/dx = f(x, y)$ under diffeomorphisms of the form $\Phi(x, y) = (\varphi(x), \psi(y))$. We generalize this problem to a system of n first order autonomous ordinary differential equations.

In this paper we present a solution to the local equivalence problem for

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n \tag{1}$$

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under the group of coordinate transformations defined by

$$\Phi(t, x_1, \cdots, x_n) = (\varphi_0(t), \varphi_1(x_1), \cdots, \varphi_1(x_1)) \tag{2}$$

This is called *the pseudo-group of web transformations*.

2 Setting the problem

Given $(U, t, x_1, \cdots, x_n)$ and $(V, T, X_1, \cdots, X_n)$ open sets with coordinates in $\mathbb{R}^{n+1}$ and ordinary differential equations (1) on U and

$$\frac{dX_i}{dT} = F_i(X_1, \cdots, X_n), \qquad i = 1, \cdots, n \tag{3}$$

on V. The usual symmetries of the ordinary differential equations are the diffeomorphisms $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which map integral curves into integral curves, that is, satisfy

$$\Phi^*(dX_i - F_i\, dT) = \sum_{j=1}^{n} a_{ij}(dx_i - f_i\, dt) \tag{4}$$

where $[a_{ij}] : \mathbb{R}^{n+1} \to \mathrm{GL}(n, \mathbb{R})$ is a smooth function. We also have the condition $\Phi^*(dT) = a\, dt$, where $a : \mathbb{R}^{n+1} \to \mathbb{R}^*$ is a smooth function. Thus, we have the following condition on Jacobian matrix of diffeomorphism Φ:

$$\Phi^*\Omega = g.\omega, \tag{5}$$

where $\Omega = (\Omega^0, \cdots, \Omega^n)^t$, $\omega = (\omega^0, \cdots, \omega^n)^t$,

$$\Omega^0 = dT,\ \ \Omega^i = dX_i - F_i\, dT,\ \ \omega^0 = dt,\ \ \omega^i = dx_i - f_i\, dt, \quad (i = 1, \cdots, n) \tag{6}$$

and $g : \mathbb{R}^{n+1} \to G$ is a smooth function, with

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \mid a \in \mathbb{R}^*,\ A \in \mathrm{GL}(n, \mathbb{R}) \right\}. \tag{7}$$

We also have the condition on the Jacobian of the diffeomorphism that

$$\Phi^*\Theta = h.\theta \tag{8}$$

where $\Theta = (\Theta^0, \cdots, \Theta^n)^t$, $\theta = (\theta^0, \cdots, \theta^n)^t$,

$$\Theta^0 = dT,\ \ \Theta^i = dX_i,\ \ \theta^0 = dt,\ \ \theta^i = dx_i, \quad (i = 1, \cdots, n), \tag{9}$$

and $h : \mathbb{R}^{n+1} \to H$ is a smooth function, with

$$H = \left\{ \begin{pmatrix} b_0 & & \bigcirc \\ & \ddots & \\ \bigcirc & & b_n \end{pmatrix} \mid b_0, \cdots, b_n \in \mathbb{R}^* \right\}. \tag{10}$$

This is an overdetermined problem in coframes ω and θ; and, we proceed the overdetermined reduction method of Proposition 9.12 of [5]. The two coframes ω and θ are connected by the $\mathrm{GL}(n+1,\mathbb{R})-$valued function A; that is

$$\omega = A\,\theta, \qquad A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -f_1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -f_n & 0 & \cdots & 1 \end{pmatrix} \tag{11}$$

The entries of matrix gAh^{-1} which are equal to

$$gAh^{-1} = \begin{pmatrix} a/b_0 & 0 & \dots & 0 \\ -(1/b_0)\sum_{i=1}^n a_{1i}f_i & a_{11}/b_1 & \dots & a_{1n}/b_n \\ \vdots & \vdots & & \vdots \\ -(1/b_0)\sum_{i=1}^n a_{ni}f_i & a_{n1}/b_1 & \dots & a_{nn}/b_n \end{pmatrix} \tag{12}$$

are zeroth-order invariants, and we can normalize them by equating gAh^{-1} to

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \tag{13}$$

This, leads to the $a = b_0$,

$$\begin{aligned} b_i &= a_{i,i} = b_0/f_i, \qquad && i = 1, \cdots, n, \\ a_{i,i-1} &= -b_0/f_{i-1}, \qquad && i = 2, \cdots, n \end{aligned} \tag{14}$$

and other a_{ij}s are zero. Furthermore, we have a family of coframes $h_0\,\theta$, with

$$h_0 = b_0 \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1/f_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/f_n \end{pmatrix}. \tag{15}$$

A fixed coframe in this family is given by $b_0 = 1$ or

$$\tilde{\theta} = \big(dt, dx_1/f_1, \cdots, dx_n/f_n\big)^t. \tag{16}$$

The elements of H which have θ as the $\tilde{\theta}$ orbit are $h = b_0 I_n$ (I_n is the $n \times n-$identity matrix). This shows that

Theorem. *The web geometry of system (1) under the pseudo-group (2) is equivalence to a equivalence problem with coframe*

$$\omega^0 = dt, \qquad \omega^i = dx_i/f_i, \quad (i = 1, \cdots, n), \tag{17}$$

and structure group $G = \{a\, I_n \mid a \in \mathbb{R}^*\}$.

3 Absorption

We lift the coframe (17) to $U \times G$ as

$$\theta^0 = a\, dt, \qquad \theta^i = a\, dx_i/f_i, \quad (i = 1, \cdots, n), \tag{18}$$

If $n = 1$, then the structure equations on $U \times G$ are

$$d\theta^0 = \alpha \wedge \theta^0, \qquad d\theta^1 = \alpha \wedge \theta^1, \tag{19}$$

and $\alpha = da/a$ is the basic Maurer-Cartan form on G; There is not any essential coefficients in (19). We must prolonged the problem to $U \times G \cong \mathbb{R}^3$, and so on. But, we can solve this problem by direct computations. Let $dx/dt = f(x)$ and $dX/dT = F(X)$ are tow given equation, and $(X, Y) = \Phi(t, x) = (\varphi_0(t), \varphi_1(x))$ be a diffeomorphism which sent $dx/dt = f(x)$ to $dX/dT = F(X)$. Then

$$F(X) = F(\varphi_1(x)) = \frac{d\varphi_1(x)}{d\varphi_0(t)} = \frac{\varphi_1'(x)}{\varphi_0'(t)}\,\frac{dx}{dt} = \frac{\varphi_1'(x)}{\varphi_0'(t)}\, f(x). \tag{20}$$

Therefore $\Phi(t, x) = (t, F^{-1}(L(x)))$, where $L(x)$ is an indefinite integral of $1/f(x)$. Thus, in this case

Theorem. *Any two first order homogeneous ODEs are web equivalence.*

Now, let $n \geq 2$, then we had deduced structure equations on $U \times G$ as

$$d\theta^0 = \alpha \wedge \theta^0, \qquad d\theta^i = \alpha \wedge \theta^i + \sum_{j=1}^{n} \frac{\ell_{ij}}{a}\, \theta^i \wedge \theta^j, \qquad (i = 1, \cdots, n). \tag{21}$$

where

$$\ell_{i,j} = f_j . \frac{\partial(\ln |f_i|)}{\partial x_j}, \qquad (i, j = 1, \cdots, n, \;\; j \neq i). \tag{22}$$

and $\alpha = da/a$ is the basic Maurer-Cartan form on G. Let us reduce the Maurer-Cartan form α back to the base manifold U by replacing them by general linear combinations of coframe elements $\alpha \longmapsto \sum_{i=0}^{n} z_i\, \theta^i$. This, leads to the system

$$d\theta^0 = \sum_{i=1}^{n} z_i\, \theta^0 \wedge \theta^i, \quad d\theta^i = \sum_{j=1}^{n} \left(\frac{\ell_{ij}}{a} - z_j\right) \theta^i \wedge \theta^j, \quad (i = 1, \cdots, n). \tag{23}$$

The combination $\ell_{ij}/a = (\ell_{ij}/a - z_j) - (z_j)$ is invariant, and so will also contribute to the essential torsion. We can normalize $\ell_{12}/a = 1$ by setting $a = \ell_{1,2}$. This normalization have the effect of eliminating all the group parameters, and so, with just one loop through the equivalence method, we have found an invariant coframe:

$$\theta^0 = dt, \qquad \theta^i = dx_i/(\ell_{12}\, f_i), \quad (i = 1, \cdots, n). \tag{24}$$

Theorem. *For $n \geq 2$, the web geometry of system (1) under the pseudo-group (2) is equivalence to $\{e\}-$equivalence problem with coframe (24), with structure equations*

$$\begin{aligned} d\theta^0 &= -\sum_{i=1}^{n} \frac{\partial \ell_{12}}{\partial x_i}\, \theta^0 \wedge \theta^i, \\ d\theta^i &= \sum_{j=1}^{n} \left(\frac{\ell_{ij}}{\ell_{12}} - \frac{\partial \ell_{12}}{\partial x_j} \right) \theta^i \wedge \theta^j, \qquad (i = 1, \cdots, n). \end{aligned} \tag{25}$$

By the theorem 8.22 of [5], we conclude that

Theorem. *For $n \geq 2$, the web symmetry group of a system (1) under the pseudo-group (2) is a finite dimensional Lie group of dimension at most $n + 1$.*

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